Complex Analysis Differential Equations X. Du

- Basic complex stuff:
  - $z = a + bi$ Conjugate: $\bar{z} = a - bi$
  - Polar form: $z = r(\cos\theta + i\sin\theta)$ Convert Cartesian to polar form
  - $a = r\cos\theta$ $b = r\sin\theta$ $r^2 = a^2 + b^2$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ $(+\pi)$
  - Denote $\cos\theta + i\sin\theta = cis\theta$
  - Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$ Proof using power series
- Multiplying and dividing complex numbers
  - Polar form
  - $z_1 = r_1 cis\theta_1$ $z_2 = r_2 cis\theta_2$
  - $z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2)$
  - Proof A:
    - $z_1 z_2 = r_1 r_2 cis(\theta_1) cis(\theta_2)$
    - $cis\theta_1 = \cos\theta_1 + i\sin\theta_1$ $cis\theta_2 = \cos\theta_2 + i\sin\theta_2$
    - $cis\theta_1 cis\theta_2 = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
    - $cis\theta_1 cis\theta_2 = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$
    - $cis\theta_1 cis\theta_2 = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$
    - $cis\theta_1 cis\theta_2 = cis(\theta_1 + \theta_2)$
    - $z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2)$
  - Proof B (trivial):
    - $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)} = r_1 r_2 cis(\theta_1 + \theta_2)$
  - Similarly: $\frac{z_1}{z_2} = \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$ Proof using the same ideas
  - Cartesian form: foil the equation
- **De Moivre's Theorem**
  - $(rcis\,\theta)^n = r^n cis(n\theta)$
  - Proof: mathematical induction using the relation $z_1 z_2 = r_1 r_2 cis(\theta_1 + \theta_2)$
- $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$
  - Compare to: $\cosh x = \frac{e^x + e^{-x}}{2}$ $\sinh x = \frac{e^x - e^{-x}}{2}$